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Approximation by Compact Operators on Certain Banach Spaces

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INTRODUCTION

This paper is concerned with the approximation properties of certain spaces of compact linear operators in the corresponding spaces of bounded linear operators. Much has been written about this in the case of operators acting on a Hilbert space. For these spaces, the compact operators have been shown to be an M-ideal in the corresponding space of bounded operators.

The concept of an M-ideal has been introduced and investigated in the fundamental paper [1] of Alfsen and Effros. According to this paper a closed subspace M of a Banach space X is an M-ideal if there is a linear projection P on the dual space X^* onto $M^\perp$, the annihilator of M, such that for every $u \in X^*$ the equality $\| u \| = \| Pu \| + \| u - Pu \|$ holds. According to [1] an important characterizing property of M-ideals is the "3 balls" property, namely: if B_i, $i = 1, 2, 3$ are open balls in X such that $B_1 \cap B_2 \cap B_3 \neq \varnothing$ and $M \cap B_i \neq \varnothing$ for $i = 1, 2, 3$, then $\bigcap_i B_i \cap M \neq \varnothing$. The approximation properties of M-ideals have been studied in [8]. We mention here that, in particular, all M-ideals are proximinal.

Due in particular to this last fact much of our attention will be focused on the question whether the space of compact operators is an M-ideal in the corresponding space of bounded linear operators.

This paper is divided into three sections. In the first section the approximation properties of $K(l_p)$, the space of compact operators on l_p in $B(l_p)$, the

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space of bounded operators, are studied for $1 \leqslant p \leqslant \infty$. Although it is known that $K(l_p)$ is an M-ideal in $B(l_p)$ for $1 < p < \infty$ and hence, as mentioned above, $K(l_p)$ must be proximinal in $B(l_p)$, we give a constructive proof of this fact. Furthermore, it is shown that $K(l_1)$ is proximinal in $B(l_1)$ and that a large class of operators on l_∞ have compact operator nearest points.

The following is an open and apparently quite difficult problem: classify those Banach spaces X and Y for which $K(X, Y)$ is a proximinal subspace (or an M-ideal) in $B(X, Y)$. Related to this question are the works of Fakhoury and Hennefeld. In [4], Fakhoury showed that $K(L_1, C(S))$ is proximinal in $B(L_1, C(S))$. In [5], Hennefeld, among other things, showed that $K(c_0)$ is an M-ideal in $B(c_0)$. Sections 2 and 3 deal with questions connected with the work of these authors.

In Section 2, the proximinality of the compact operators from L_1 into any separable uniformly rotund space in the corresponding space of bounded operators is established, whereas, in Section 3, the compact operators with range in certain spaces of continuous functions is seen to be an M-ideal in the corresponding space of bounded operators.

At this time we would like to thank Professor I. D. Berg for his helpful suggestions related to Section 1.

Throughout this article, $B(x, R)$ will denote the open ball centered at x having radius R. The metric projection of a vector x onto a subspace M will be indicated by $P_M(x)$. $B(X, Y)$ (resp. $B(X)$) will designate the space of bounded linear operators mapping the Banach space X into a Banach space Y (resp. X into X) while $K(X, Y)$ (resp. $K(X)$) will denote the corresponding space of compact operators. The restriction of an operator S to a subspace V will be given by $S \mid V$ and $\bigvee (v_1, ..., v_n)$ will denote the linear span of v_i, $i = 1, ..., n$. Finally, two vectors in l_p will be called orthogonal, if they have disjoint supports.

1. Compact Operator Approximation in $B(l_p)$

In this section, certain theorems on compact operator approximation in $B(l_p)$ are proved. Special emphasis is placed on the approximation properties of the compact operators for $p = 1$ or ∞ since in these cases the compact operators are not M-ideals [9, Theorem 6.2]. However, many approximation properties analogous to the case $1 < p < \infty$ are seen to still hold.

Several authors have studied compact operator approximation in $B(l_2)$, for example [3, 6, 7]. However, not much has been discussed for the case $p \neq 2$. It is known that for $1 < p < \infty$, $K(l_p)$ is proximinal in $B(l_p)$. This follows from the fact that $K(l_p)$ is an M-ideal in $B(l_p)$ and that all M-ideals are proximinal subspaces. The proof, as given in [1], is very nonconstructive. There is a simple constructive proof that the compact operators are proximinal in $B(l_2)$ but this depends on spectral theory and the polar decomposition of an operator. Both tools are unavailable in $B(l_p)$ for general p. Nevertheless the following theorem, although known, provides a new and constructive proof of the fact that $K(l_p)$ is proximinal in $B(l_p)$.

THEOREM 1.1. $K(l_p)$ is a proximinal subspace in $B(l_p)$ for $1 < p < \infty$.

Proof. Let $T \in B(l_p)\backslash K(l_p)$ for any fixed p, $1 < p < \infty$ and set $R \equiv d(T, K(l_p))$, $R > 0$. Without loss of generality one may assume that T is a tri-block-diagonal operator with respect to the canonical basis $\{e_i\}_{i=1}^{\infty}$. To see this, note that for any positive sequence $\{\delta_i\}_{i=1}^{\infty}$ satisfying $\sum_{i=1}^{\infty} \delta_i = \delta$, the basis $\{e_i\}_{i=1}^{\infty}$ may be divided into a sequence of adjacent finite blocks increasing in length so rapidly that the spaces H_i, say, spanned by successive blocks of the e_n satisfy

$$\| P_L T P_{H_i} \| < \delta_i \qquad \text{and} \qquad \| P_{H_i} T P_L \| < \delta_i$$

so long as the space L is perpendicular to P_{H_i}, $P_{H_{i-1}}$, and $P_{H_{i+1}}$, where P_V denotes the orthogonal projection onto V. Now note that T minus the tridiagonal part of the operator matrix, call it $\hat{T}$, is a compact operator of norm less than or equal to δ. Thus attention may now be focused on $\hat{T}$ as it is a compact perturbation of T.

The operator S defined by $S(e_n) = a_n \hat{T}(e_n)$ for sufficiently slowly increasing scalars a_n, $a_n \to 1$ provides the desired $\hat{T} + K$. The sequence $\{a_i\}_{i=1}^{\infty}$ will be defined by induction. Pick $a_1 > 0$ to satisfy

$$\| a_1 \hat{T} \| < R = d(\hat{T}, K(l_p)).$$

Now assume n steps have been completed. The partially constructed operator has the form

$$S_N \equiv \sum_{i=1}^{N} a_i P_{E_i^\perp} \hat{T} P_{E_i^\perp},$$

where $E_i^\perp$ is $\bigvee \{e_{n(i)}, e_{n(i)+1}, \ldots, \ldots\}$ for some appropriate $n(i)$. For sufficiently large k, it is easily seen that

$$S_N |_{E_k^\perp} = \left(\sum_{i=1}^{N} a_i \hat{T} \right) \Big|_{E_k^\perp},$$

Now by construction $\| S_N \| < d(T, K(l_p))$. To proceed with the induction, select a δ_{N+1} so that

$$\| S_N \| + \delta_{N+1} < d(T, K(l_p)).$$

Pick E_{N+1} in the following way: Without loss of generality, assume

$$\| S_N |_{E_N^\perp} \| < \sum_{i=1}^{N} a_i R + \delta_{N+1}/2$$

(otherwise one may enlarge E_N until the above inequality is satisfied). To E_N attach so many blocks of finite-dimensional subspaces $A_{i(N)}, A_{i(N)+1}, \ldots, A_{i(N)+l}$ that for any unit vector v the projection of v onto some consecutive pair of the A_i is no larger than $\delta_{N+1}/2$. (The consecutive pair of the A_i of course depends on the vector v.) Now define

$$E_{N+1} \equiv \bigvee (E_N \cup A_{i(N)} \cup \cdots \cup A_{i(N)+l}).$$

The operator

$$S_{N+1} = S_N + a_{N+1} P_{E_{N+1}^\perp} T P_{E_{N+1}^\perp}$$

is now defined where a_{N+1} may now be chosen as, for example, $1 - \sum_{i=1}^{N} a_i - \delta_{N+1}/2$. Any unit vector v may be split into the form $v_1 + v_2 + v_3$, where v_2 is in some consecutive pair of the A_i (call them A_s, A_{s+1}) with $\| v_2 \| < \delta_{N+1}/2$, $v_1 \in \bigvee_{i<s}(A_i)$, and $v_3 \in \bigvee_{i>s+1}(A_i)$. It is easy to check that since v_1 is orthogonal to v_3, $S_{N+1}(v_1)$ is orthogonal to $S_{N+1}(v_3)$, and $S_{N+1}(v_2)$ is small, then

$$\| S_{N+1} \| < d(T, K(l_p)).$$

It is evident from the construction that the a_i may be chosen so that $\sum_{i=1}^{\infty} a_i = 1$ and hence

$$S = \sum_{i=1}^{\infty} a_i P_{E_i^\perp} \hat{T} P_{E_i^\perp}$$

is the required operator. This completes the proof.

It was shown in [8], that for an infinite-dimensional M-ideal M and for $x \notin M$, then $P_M(x)$ was not compact. The following shows that a stronger assertion holds in $B(l_p)$.

COROLLARY 1.2. *For* $T \in B(l_p) \backslash K(l_p)$, $P_{K(l_p)}(T)$ *is not strong operator compact.*

Proof. Let E_n be a sequence of finite rank projections converging strongly to the identity I and suppose that K is a best compact operator approximant to T. Then evidently for each n,

$$TE_n - K(I - E_n)$$

is also a best compact operator approximant to T and

$$TE_n - K(I - E_n) \rightharpoonup T \qquad \text{since} \quad K(I - E_n) \rightharpoonup 0,$$

where $\rightharpoonup$ indicates convergence in the strong operator topology.

We now turn our attention to $K(l_1)$ in $B(l_1)$. It is easy to see that the proof of Theorem 1 is invalid for $p = 1$ or ∞ and so the question of the proximinality of $K(l_1)$ in $B(l_1)$ is not immediately resolved. Nevertheless the following may still be proved.

THEOREM 1.3. $K(l_1)$ is a proximinal subspace of $B(l_1)$.

Proof. As shown in [10, p. 220], every operator on l_1 has a matricial representation with respect to the canonical basis $\{e_i\}_{i=1}^{\infty}$ and that

$$\text{(a)} \quad C \in K(l_1) \qquad \text{iff} \quad \lim_{n\to\infty} \sup_j \sum_{i=n}^{\infty} | c_{ij} | \to 0;$$

$$\text{(b)} \quad T \in B(l_1) \qquad \text{implies} \quad \| T \| = \sup_j \sum_{i=1}^{\infty} | t_{ij} |.$$

It will now be shown that $d(T, K(l_1)) = \lim_{n\to\infty} \sup_j \sum_{i=n}^{\infty} | t_{ij} | \equiv R$. Evidently, $d(T, K(l_1)) \geqslant R$ since for all $C \in K(l_1)$

$$\| T - C \| \geqslant \lim_{n\to\infty} \sup_j \sum_{i=n}^{\infty} | t_{ij} - c_{ij} |$$

$$= \lim_{n\to\infty} \sup_j \sum_{i=n}^{\infty} | t_{ij} | = R.$$

To prove the claim and the theorem, a compact operator of distance R from T will be produced. Now for fixed j,

$$\text{if} \quad \sum_{i=1}^{\infty} | t_{ij} | \leqslant R, \qquad \text{set} \quad c_{ij} = 0 \quad i = 1,..., \infty$$

$$\text{if} \quad \sum_{i=1}^{\infty} | t_{ij} | > R, \qquad \text{set} \quad c_{ij} = t_{ij} \quad i = 1,..., n,$$
$$c_{ij} = 0 \quad i > n,$$

where n is chosen so that

$$\sum_{i=1}^{\infty} | t_{ij} - c_{ij} | = R. \tag{1.1}$$

(Note: In certain cases, the final nonzero c_{ij} might be defined as at_{ij}, $0 < a < 1$ instead of t_{ij} in order to satisfy (1.1)). Clearly,

$$\| T - C \| = \sup_j \sum_{i=1}^{\infty} | t_{ij} - c_{ij} | = R.$$

It remains to show that $C \in K(l_1)$. If $C \notin K(l_1)$ then

$$\lim_{n\to\infty} \sup_j \sum_{i=n}^{\infty} | c_{ij} | \geqslant \epsilon \qquad \text{for some} \quad \epsilon > 0.$$

Now pick N_0 so that

$$\sup_j \sum_{i=N_0}^{\infty} | t_{ij} | \leqslant R + \epsilon/2 \tag{1.2}$$

and select a j_0 satisfying $\sum_{i=N_0}^{\infty} | c_{ij_0} | \geqslant 3\epsilon/4$. For this $j_0 \sum_{i=N_0}^{\infty} | t_{ij_0} | \geqslant R + 3\epsilon/4$ which contradicts (1.2). This completes the proof.

Although not an M-ideal in $B(l_1)$, $K(l_1)$ shares similar approximation properties with M-ideals. The next proposition should be contrasted with Theorem 3 in [8].

PROPOSITION 1.4. *Let $T \in B(l_1)\backslash K(l_1)$. Then the set of best compact operator approximants is infinite dimensional.*

We omit the proof. It is clear from the proof of Theorem 1.3 that there are many ways to alter the compact operator best approximant. However, in contrast with the M-ideal case where it is known that for $x \in X\backslash M$ (here M is an M-ideal and X is the ambient Banach space), span $Py(x) = M$, the above proposition gives in general the strongest result as the next example demonstrates.

EXAMPLE 1.5. Let $C \in K(l_1)$ be the compact operator defined as

$$\begin{aligned} c_{1j} &= 1, \qquad j = 1,..., \infty, \\ c_{ij} &= 0, \qquad \text{otherwise}. \end{aligned}$$

Then $d(C, \text{span } P_{K(l_1)}(I)) \geqslant 1$.

Proof. From the formulas used in Theorem 1.3, it is easily checked that if $C \in P_{K(l_1)}(I)$ then $\lim_{j\to\infty} \sum_{i=1}^{\infty} | c_{ij} | \to 0$. Now for $C \in P_{K(l_1)}(I)$, $j = 1,..., n$ then $d(C, \sum_{i=1}^{n} a_i C_i) \geqslant 1$ for fixed a_i $i = 1,..., n$ and thus by the continuity of $d(C, \cdot)$ the conclusion follows.

As a final remark, the following proposition should be mentioned.

PROPOSITION 1.6. *There is a continuous homogeneous metric selection for $P_{K(l_1)}(\cdot)$.*

Again, the proof is omitted. It follows the familiar pattern of establishing that $P_{K(l_1)}(\cdot)$ is a lower semi-continuous set-valued mapping and then appealing to Michael's selection theorem.

We now consider the case of $B(l_\infty)$. As mentioned in [10, p. 220] not every bounded linear operator on l_∞ has a matricial representation. This is due to the fact that l_∞ does not have a basis. This leads to problems for compact operator approximation. In particular, it appears that it is unknown whether $K(l_\infty)$ is proximinal in $B(l_\infty)$. However a large class of operators in $B(l_\infty)$ do have a matricial representation, namely, those operators which are adjoints of operators in $B(l_1)$, hereafter denoted $[B(l_1)]' \subset B(l_\infty)$. For these operators, the following holds

THEOREM 1.7. *Let $T \in [B(l_1)]'$. Then T has a best compact operator approximant.*

Proof. As shown in [10, p. 220], operators with such a matricial representations have the properties that $\| T \| = \sup_i \sum_{j=1}^{\infty} | t_{ij} |$ and that T is compact iff $\lim_{n\to\infty} \sup_i \sum_{j=n}^{\infty} | t_{ij} | = 0$. Thus if $d(T, K(l_\infty)) = d(T, [K(l_1)]')$ then a proof analogous to Theorem 1.3 would allow one to construct a best approximant. The proof then is the same except column operations there are now replaced by row operations. We now show that $d(T, K(l_\infty)) = d(T, [K(l_1)]')$.

For all $C \in K(l_\infty)$, define $C \mid c_0$ as C'. The following proposition will be established, namely that

$$\| C' |_{E_n^\perp} \| \to 0 \qquad \text{as} \quad n \to \infty.$$

(Here, $E_n = \bigvee (e_1, \ldots, e_n)$.) Now suppose $\| C \mid E_n^\perp \| \geqslant 1$ for all n. Pick $e_1' \in E_{n1}$ such that $\| C'(e_1') \| \geqslant \frac{1}{2}$. It will now be shown that $P_{E_{n1}} C' |_{E_m^\perp} \to 0$ as $m \to \infty$. If this were not true one could pick a large finite set of v_i having disjoint support and projecting back to E_{n1} in such a manner that $\sum_{i=1}^{n} v_i$ would have huge norm. Now pick e_2' orthogonal to e_1' and such that

$$\| P_{E_{n1}} C' e_2 \| \approx 0 \qquad \text{and} \qquad \| C' e_2 \| \geqslant \tfrac{1}{2}.$$

Note that $d(C'e_1, C'e_2) \geqslant \frac{1}{2}$. By continuing this process we may contradict the compactness of C'. Hence $\| C' |_{E_n^\perp} \| \to 0$. This shows that

$$d(T, K(l_\infty)) \geqslant \lim_{n\to\infty} \| T |_{E_n^\perp} \| = d(T, [K(l_1)]')$$

and our proof is complete.

As mentioned earlier it is known that $K(l_\infty)$ is not an M-ideal in $B(l_\infty)$. However just as in the $B(l_1)$ case, for operators in $B(l_\infty)$ with matricial representation, the corresponding set of best compact approximants satisfy many of the same properties as compact approximants in $B(l_p)$, $1 < p < \infty$.

2. Compact Operator Approximation of Operators on L_1

Let $(S, \sum, \mu)$ be a σ-finite positive measure space. In this section we investigate operators on $L_1(S, \sum, \mu)$ with range in a separable uniformly rotund Banach space X. The main result of this section is the fact that every bounded linear operator on $L_1(S, \sum, \mu)$ into X has an element of best approximation from $K(L_1(S, \sum, \mu), X)$.

Definition. Let B be a bounded subset in a Banach space X. The Kuratowski measure of noncompactness $\alpha(B)$ of B is the greatest lower bound of all $\alpha > 0$ such that there is an α-net of B (i.e., points $x_i \in X$, $i = 1,..., n$, such that the balls $B(x_i, \alpha)$ cover B).

Let B be a bounded set in X. The next lemma shows that, if for $m > n$, A_n and A_m are finite $\alpha(B) + 1/n$ and $\alpha(B) + 1/m$-nets of B, respectively, then A_m can be chosen "close" to A_n.

Lemma 2.1. *Let X be a uniformly rotound Banach space. Then for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that for every $m > n$ there is an $\alpha(B) + 1/m$-net A_m of B with $d(A_n, A_m) < \epsilon$, where d is the Hausdorff metric.*

Proof. For $x \in A_n$, $y \in A_m$ denote $I(x, y) = B(x, \alpha(B) + 1/n) \cap B(y, \alpha(B) + 1/m) \cap B$. We show that, given $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that for any fixed $m > n$ and for every $x \in A_n$, $y \in A_m$ there is a $y' \in B(x, \epsilon)$ with $I(x, y) \subset B(y', \alpha(B) + 1/m)$. Clearly the set of all such points y' forms a finite $\alpha(B) + 1/m$-net of B with the required property.

Suppose there is an $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there is an $x_n \in A_n$ and a $y_n \in A_m$, $m > n$, with

$$I(x_n, y_n)\backslash B(y, \alpha(B) + 1/m) \neq \varnothing$$

for every $y \in B(x_n, \epsilon_0)$. Clearly

$$\| x_n - y_n \| \geqslant \epsilon_0 \tag{2.1}$$

for each $n \in \mathbb{N}$. Put $b_n = \| y_n - x_n \|$, $z_n = (1 - \epsilon_0/2b_n) x_n + (\epsilon_0/2b_n) y_n$. We have

$$\| x_n - z_n \| = \epsilon_0/2. \tag{2.2}$$

By assumption, for every $n \in \mathbb{N}$, there is a $z_n' \in I(x_n, y_n)\backslash B(z_n, \alpha(B) + 1/m)$. Hence there are subsequences x_k, y_k, z_k, and z_k' such that $\lim \| x_k - z_k' \| \leqslant \alpha(B)$, $\lim \| y_k - z_k' \| \leqslant \alpha(B)$ and $\lim \| z_k - z_k' \| \geqslant \alpha(B)$. It can be easily shown that this together with (2.2) implies $\lim \| \frac{1}{2}(x_k + y_k - 2z_k')\| \geqslant \alpha(B)$, which together with (2.1) contradicts the uniform rotundity of X.

THEOREM 2.2. *Let B be a bounded set in a uniformly rotund Banach space X. Then there is a compact set K in X such that for every $x \in B$ we have* $\operatorname{dist}(x, K) \leqslant \alpha(B)$.

Proof. According to Lemma 2.1 find an $n_1 \in \mathbb{N}$ such that for any $m > n_1$ there is an A_m with $d(A_{n_1}, A_m) < \frac{1}{2}$. Put $B_1 = A_{n_1}$. Suppose that $B_k = A_{n_k}$ with $d(A_m, A_{n_k}) < \frac{1}{2}^k$ for every $m > n_k$ has been constructed. Find an n_{k+1} such that $d(A_{n_k}, A_{n_{k+1}}) < \frac{1}{2}^k$ and such that for every $m > n_{k+1}$ there is an A_m with $d(A_m, A_{n_{k+1}}) < \frac{1}{2}^{k+1}$. Put $B_{k+1} = A_{n_{k+1}}$. The set $K = \operatorname{cl} \bigcup_{k \in \mathbb{N}} B_k$ has obviously the required properties. Indeed, $\operatorname{dist}(x, K) \leqslant \alpha(B)$ for any $x \in B$. Further, for every $k \in \mathbb{N}$ the set $B_1 \cup B_2 \cup \cdots \cup B_{k+1}$ is a finite $\frac{1}{2}^k$-net of K. It follows that K is compact.

THEOREM 2.3. *Let $L_1 = L_1(S, \sum, \mu)$, where $(S, \sum, \mu)$ is a σ-finite positive measure space. If X is a separable uniformly rotound Banach space, then $K(L_1, X)$ is proximinal in $B(L_1, X)$.*

Proof. Since $B(L_1, X) = W(L_1, X)$, $W(L_1, X)$ the corresponding space of weakly compact operators, if X is a reflexive Banach space, we have only to show that $K(L_1, X)$ is proximinal in $W(L_1, X)$.

Let $T \in W(L_1, X)$. Then, by the representation theorem VI.8.10 [2], there exists a μ-essentially unique bounded measurable function $x(t)$ on S into a weakly compact subset B of X such that $\| T \| = \operatorname{ess\,sup}_{s \in S} \| x(s) \|$. Let $\alpha(B)$ be the Kuratowski measure of noncompactness of B. Obviously we have for any compact set $K \subset X$

$$\sup_{s \in S} \operatorname{dist}(x(s), K) \geqslant \alpha(B). \tag{2.3}$$

Let $L \in K(L_1, X)$. By Corollary VI. 8. 11 [2] there is a μ-null set $E \subset S$ and a compact set K_1 such that for the corresponding function $y : S \to X$ we have $y(s) \in K_1$ for every $s \in S \backslash E$. Hence, by (2.3), we have

$$\begin{aligned} \| T - L \| &= \operatorname*{ess\,sup}_{s \in S} \| x(s) - y(s) \| \\ &= \sup_{s \in S} \| x(s) - y(s) \| \geqslant \sup_{s \in S \backslash E} \| x(s) - y(s) \| \\ &\geqslant \sup_{s \in S} \operatorname{dist}(x(s), K_1) \geqslant \alpha(B). \end{aligned} \tag{2.4}$$

According to Theorem 2.1 construct a compact set K_2 such that for any $s \in S$, $\operatorname{dist}(x(s), K_2) \leqslant \alpha(B)$. For each $s \in S$ find the unique $k(s) \in \operatorname{conv} K_2$ with $\| x(s) - k(s) \| = \operatorname{dist}(x(s), \operatorname{conv} K_2) \leqslant \alpha(B)$. It can be easily shown that this function $k : S \to \operatorname{conv} K_2$ is μ-measurable. Let L_0 be the corresponding compact operator (Corollary VI.8.11 [2]). Then we have

$$\| T - L_0 \| = \operatorname*{ess\,sup}_{s \in S} \| x(s) - k(s) \| \leqslant \alpha(B),$$

which, together with (2.4), shows that L_0 is an element of best approximation of T.

3. Compact Operators with Range in $C(S)$

Let S be a compact Hausdorff space, R a closed subspace of S, Y a Banach space. We denote by $C(S \| R, Y)$ the space of all continuous functions on S with values in Y, vanishing on R. If $Y = \mathbb{R}$, we use the notation $C(S \| R)$. Let X be a Banach space. In this section we investigate the following question: under which assumptions is $K(X, C(S \| R))$ an M-ideal in $B(X, C(S \| R))$? We give a sufficient condition and show that, for some Banach spaces X, this condition is the best we can expect. As a consequence of this result we obtain that compact operators on $C[0, 1]$ into itself are not an M-ideal in the space of bounded operators on $C[0, 1]$ into itself.

Theorem 3.1. *If R is the set of all accumulation points of S, then $K(X, C(S \| R))$ is an M-ideal in $B(X, C(S \| R))$ for an arbitrary Banach space X.*

Proof. According to the representation theorem VI.7.1 [2] $B(X, C(S \| R))$ is isometrically isomorphic to the space $C_{w^*}(S \| R, X^*)$ of all w^*-continuous functions $u : S \to X^*$ vanishing at R, equipped with the supremum norm, and $K(X, C(S \| R))$ is isometrically isomorphic to $C(S \| R, X^*)$. We show that $C(S \| R, X^*)$ has the 3-balls property in $C_{w^*}(S \| R, X^*)$.

Let $B(x_i, r_i)$, $i = 1, 2, 3$, be open balls in $C_{w^*}(S \| R, X^*)$ such that there is an $x_0 \in C_{w^*}(S \| R, X^*)$ with $\| x_i - w_0 \| < r_i$, and such that $B_i \cap C(S \| R, X^*) \neq \varnothing$, $i = 1, 2, 3$. Then there is an $\epsilon > 0$ such that

$$r_i > \operatorname{dist}(x_i, C(S \| R, X^*)) + 2\epsilon. \tag{3.1}$$

for $i = 1, 2, 3$. Choose $y_i \in C(S \| R, X^*)$ such that

$$\| x_i - y_i \| < \operatorname{dist}(x_i, C(S \| R, X^*)) + \epsilon.$$

For every $s \in R$ there is a neighborhood $U(s)$ such that $\sup_{t \in U(s)} \| y_i(t) \| < \epsilon$, $i = 1, 2, 3$. Denoting $V = \bigcup_{s \in R} U(s)$, we have for $i = 1, 2, 3$

$$\begin{aligned} \sup_{s \in V} \| x_i(s) \| &\leqslant \| x_i - y_i \| + \sup_{s \in V} \| y_i(s) \| \\ &< \operatorname{dist}(x_i, C(S \| R, X^*)) + 2\epsilon < r_i . \end{aligned}$$

Define

$$y_0 = x_0 \qquad \text{on } S\backslash V$$
$$= 0 \qquad \text{on } V.$$

Then $y_0 \in C(S \parallel R, X^*)$ and, by (3.1), (3.2), we have for $i = 1, 2, 3$

$$\| x_i - y_0 \| = \max(\sup_{s \in S\backslash V} \| x_i(s) - x_0(s)\|, \sup_{s \in V} \| x_i(s)\|) < r_i .$$

Thus $C(S \parallel R, X^*)$ has the 3-balls property in $C_{w^*}(S \parallel R, X^*)$.

THEOREM 3.2. *Let a Banach space X have the following property: There is an $\epsilon_0 > 0$, $\delta_0 > 0$, $v_0 \in X^*$, $\| v_0 \| < 1$, $u_0 \in X^*$, $\| u_0 \| = 1$, and a sequence $\{u_n\}$ in X^* such that*

(i) $\lim \| u_n \| = 1$,
(ii) $w^*\text{-}\lim u_n = 0$,
(iii) $\| u_n - v_0 \| \leqslant 1 - \delta_0$ *for* $n \in \mathbb{N}$,
(iv) $\lim \|(2 - \epsilon_0) u_0 + u_n \| \geqslant 2 + \epsilon_0$.

Let there be an accumulation point of a metrizable S which is not in R. Then $K(X, C(S \parallel R))$ has not the 2-balls property, consequently, it is not an M-ideal in $B(X, C(S \parallel R))$.

Proof. Let r_0 be an accumulation point of S, $r_0 \notin R$. Then there are two disjoint sequences $\{a_n\}$, $\{t_n\}$ consisting of pairwise different points both converging to r_0. Denote $M = \text{cl}\{t_n\}$. It is easily seen that it is possible to construct a sequence $\{U_n\}$ of pairwise disjoint open sets such that $s_n \in U_n$ and $U_n \cap M = \varnothing$ for every $n \in \mathbb{N}$. By Urysohn's lemma there is, for every $n \in \mathbb{N}$ a continuous function f_n, $0 \leqslant f_n \leqslant 1$, such that $f_n(s_n) = 1$ and $f_n = 0$ on $S\backslash U_n$, and a continuous function f_0, $0 \leqslant f_0 \leqslant 1$, with $f_0 = 1$ on $M \cup \text{cl}\{s_n\}$ and $f_0 = 0$ on R. Choose ϵ_0, δ_0, $v_0 \in X^*$, $u_0 \in X^*$, and $\{u_n\} \subset X^*$ such that (i)–(iv) are fulfilled. Put

$$x_0(s) = \sum_{n=1}^{\infty} f_n(s)\, u_n$$

and define

$$x_1(s) = f_0(s)\, x_0(s),$$
$$x_2(s) = f_0(s)((2 - \epsilon_0)\, u_0 + x_0(s)).$$

Define further

$$x_3(s) = f_0(s)((1 - \epsilon_0/2)\, u_0 + x_0(s)),$$
$$x_1(s) = f_0(s)\, v_0\,,$$
$$x_1(s) = f_c(s)((2 - \epsilon_c)\, u_0 + v_0).$$

It is easy to see that $x_3 \in B(x_1, 1) \cap B(x_2, 1)$, $x_1 \in C(S \,\|\, R, X^*) \cap B(x_1, 1)$, $x_1 \in C(S \,\|\, R, X^*) \cap B(x_2, 1)$. We show that there is no function from $C(S \,\|\, R, X^*)$ in $B(x_1, 1) \cap B(x_2, 1)$. Let $x \in B(x_1, 1) \cap B(x_2, 1)$. Then $\| x(s_n) - (2 - \epsilon_0)\, u_0 - u_n \| < 1$. Hence $\|(2 - \epsilon_0)\, u_0 + u_n \| - \| x(s_n)\| < 1$ which, together with condition (iv), implies $\limsup \| x(s_n)\| \geqslant 1 + \epsilon_0$. On the other hand we must have $\| x(t_n)\| < 1$ for every $n \in \mathbb{N}$. Hence x cannot be continuous at r_0. Thus $C(S \,\|\, R, X^*)$ has not the 2-balls property in $C_{w^*}(S \,\|\, R, X^*)$. According to the representation theorem VI.7.1 [2] the same is true for $K(X, C(S \,\|\, R))$ in $B(X, C(S \,\|\, R))$.

Remark. It may be easily verified that $C[0, 1]$ and l_1 fulfil conditions (i)–(iv) of Theorem 3.2. For $C[0, 1]$ take, e.g.,

$$u_n(t) = \tfrac{1}{2}t - (i - 1)/2n, \qquad t \in [(i - 1)/n, i/n),\ i = 1,..., n,\ n \in \mathbb{N},$$
$$u_n(1) = \tfrac{1}{2}n, \qquad n \in \mathbb{N},$$
$$v_0(t) = \tfrac{1}{2}t, \qquad t \in [0, 1],$$
$$u_0(t) = t, \qquad t \in [0, 1].$$

The l_1 case is left to the reader.

COROLLARY 3.3. $K(C[0, 1], C[0, 1])$ and $K(l_1, C[0, 1])$ are not M-ideals in the corresponding spaces of bounded operators.

4. OPEN PROBLEMS

During the course of these investigations certain problems arose some of which have already been mentioned in this paper. We mention these questions again along with a few others.

First, is $K(l_\infty)$ proximinal in $B(l_\infty)$? As seen earlier, a certain subclass of operators in $B(l_\infty)$ admitted best compact operator approximants but the general question still appears open. If true, this would mean that the compact operators were proximinal for all p, $1 \leqslant p \leqslant \infty$.

In the case of L_p the question of best compact operator approximation appears again to be open. Are the compact operators proximinal in the space of bounded linear operators on L_p? More generally, is $K(L_p)$ an M-ideal in $B(L_p)$?

In the case of the compact operators on $C(S)$, again many questions arise. Is $K(C(S))$ proximinal in $B(C(S))$? Is there a reasonable distance formula for $\text{dist}(T, C(S))$ if $T \in B(C(S)) \backslash K(C(S))$?

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